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Approximate analyses of Fourier and non-Fourier heat conduction models by the variational principles based on Laplace transforms

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ABSTRACT
Approximate analysis is a major application of variational principles for heat conduction. Recently, O'Toole's variational principle for Fourier's law has been extended to non-Fourier heat conduction models, which are applied to approximate analyses based on the Rayleigh–Ritz method. Suitable trial functions satisfying boundary conditions are sought, and then substituted into the variational principles to obtain the undetermined coefficients. From the inverse Laplace transforms, the approximate solutions are obtained. Examples are provided for 1D problems for different heat conduction models. The largest calculation errors are one or two orders of magnitude smaller than the equilibrium temperature, which will tend to be zero.

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1. Introduction

Great efforts have been made on approximate methods for heat conduction problems [1–4]. Although the thermal balance method is one of the most typical and commonly used methods [5–8], variational principles for heat conduction can also provide approximate analytical methods, which may help develop numerical methods and discuss the characteristics of solutions [9,10]. There are several variational principles for dissipative processes including heat conduction, which have been pioneered by Onsager et al. [11–13], Prigogine et al. [14], Biot [15,16], Gyarmati et al. [17,18], and others [19–23]. Biot's variational principle is one of the most frequently used methods for approximate analyses [24–27]. Gyarmati's variational principle of dissipative processes, which is usually called the Governing Principle of Dissipative Processes (GPDP), is also often applied to approximate analyses [28–31]. Both Biot's method and GPDP are mainly used for approximate analyses in Fourier heat conduction, whose results have been enriched for different problems. However, for non-Fourier heat conduction, which must be considered at low temperature, high heat flux, and supertransient and small-scale processes [32–37], the variational analytical method is rarely studied.

O'Toole [38] first used Laplace transforms to provide variational principles for time-dependent transport processes with the Laplace transform [39] $U(x, y, z, p)$ of a function $u(x, y, z, t)$ expressed as $U(x,y,z,p) = \int_0^{+\infty} u(x,y,z,t)e^{-pt}dt$. O'Toole's variational principle is only for Fourier's law with the first type of boundary condition, which specifies the boundary temperature. Recently, O'Toole's method has been generalized to other cases, including non-Fourier heat conduction and all three types of boundary conditions [40]. Variational principles based on Laplace transforms are proposed for several non-Fourier heat conduction models, including the Cattaneo–Vernotte (CV) model [40,41], the Jeffrey model [42], the two-temperature (TT) model [43], and the Guyer–Krumhansl (GK) model [44].

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Nomenclature

a	thermal diffusivity in Fourier's law, m^2/s	T	temperature, K
c	isothermal first-sound velocity in the GK model, m/s	T_e	electron temperature in the TT model, K
C_E	heat wave velocity, m/s	x	space coordinate, m
c_V	specific heat, $J/K \cdot kg$	α_e	thermal diffusivity of the electrons in the TT model, m^2/s
D	spatial domain	α_E	equivalent thermal diffusivity in the TT model, m^2/s
F	Laplace transform of temperature, $F = \int_0^{+\infty} Te^{-pt}dt$	λ	thermal conductivity in Fourier's law, $W/K \cdot m$
Fo	Fourier number	ρ	mass density, kg/m^3
k	total thermal conductivity in the Jeffrey model, $W/K \cdot m$	τ	thermal relaxation time in the CV and Jeffrey models, s
k_F	thermal conductivity for Fourier heat conduction in the Jeffrey model, $W/K \cdot m$	τ_N	single-phonon relaxation time for normal processes in the GK model, s
l	boundary coordinate, m	τ_R	momentum loss relaxation time in the GK model, s
q	heat flux, W/m^2		
t	time coordinate, s		

This paper applies the variational principles based on Laplace transforms to approximate the method for Fourier and non-Fourier heat conductions. The approximate analyses can be considered as an extension of the Rayleigh–Ritz variation method. First, in the Laplace transform space, suitable expressions of the trial functions satisfying all boundary conditions are sought, which have undetermined coefficients. Then, these trial functions are substituted into the variational principles based on Laplace transforms to obtain the undetermined coefficients. After determining the coefficients and trial functions, approximate solutions can be derived from the inverse Laplace transforms of the trial functions. Approximate analytical examples are provided and discussed for one-dimensional problems with the first type of boundary condition, and different heat conduction models, including Fourier's law, the CV model, the Jeffrey model, the TT model, and the GK model.

2. Approximate method and examples

2.1. Fourier's law

Fourier's law and the energy conservation equation are expressed as

$$q + \lambda \nabla T = 0, \tag{1}$$

$$\nabla \cdot q = -\rho c_V \frac{\partial T}{\partial t}. \tag{2}$$

The thermal conductivity λ, the specific heat c_V, and particularly the mass density ρ are positive and constant in time and space. Eqs (1) and (2) can be combined to give the heat conduction equation:

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c_V} \nabla^2 T. \tag{3}$$

For the first type of boundary condition, the variational principle based on Laplace transforms [40] is

$$\delta \left\{ \iiint_D \left[\frac{\lambda}{\rho c_V} |\nabla F|^2 + p|F|^2 - 2F\left(T|_{t=0}\right) \right] dV \right\} = 0, \tag{4}$$

Consider a one-dimensional problem in $0 \le x \le l$, where the initial condition is taken as $T|_{t=0} = T_0\left(1 + \sin\frac{\pi x}{l}\right)$ and the boundary conditions are taken as $T|_{x=0,\, l} = T_0$. The classical solution of this problem is

$$T(x,t) = T_0\left(1 + e^{-\frac{\lambda \pi^2 t}{\rho c_V l^2}} \sin\frac{\pi x}{l}\right). \tag{5}$$

The approximate solution of this problem is written as $T_1^*(x,t)$. F_1^* is the Laplace transform of $T_1^*(x,t)$, which should satisfy Eq. (4) and the boundary conditions. Because there are two boundary conditions and one variational equation, F_1^* can be expressed as a second-order polynomial approximation:

$$F_1^* = k_1x^2 + b_1x + c_1. \tag{6}$$

The Laplace transforms of the boundary conditions are

$$F_1^*\big|_{x=0,l} = \frac{T_0}{p}. \tag{7}$$

From Eqs. (6) and (7), we can obtain

$$c_1 = \frac{T_0}{p}, \tag{8-a}$$

$$k_1l + b_1 = 0. \tag{8-b}$$

Then, we have a second-order polynomial approximation expressed by k_1:

$$F_1^* = k_1x^2 - k_1lx + \frac{T_0}{p}. \tag{9}$$

Substituting Eq. (9) into Eq. (4) leads to

$$\begin{aligned}\delta\Bigg\{\int_0^l\Bigg[&\frac{\lambda}{\rho c_V}(2k_1x-k_1l)^2 + p\left(k_1x^2-k_1lx+\frac{T_0}{p}\right)^2\\ &-2\left(k_1x^2-k_1lx+\frac{T_0}{p}\right)T_0\left(1+\sin\frac{\pi x}{l}\right)\Bigg]dx\Bigg\} = 0.\end{aligned} \tag{10}$$

Equation (10) could be simplified to

$$\delta\left[\left(\frac{\lambda l^3}{3\rho c_V}+\frac{pl^5}{30}\right)k_1^2 + \frac{8T_0l^3k_1}{\pi^3}\right] = \left[2\left(\frac{\lambda l^3}{3\rho c_V}+\frac{pl^5}{30}\right)k_1 + \frac{8T_0l^3}{\pi^3}\right]\delta(k_1) = 0. \tag{11}$$

From Eq. (11),

$$k_1 = \frac{-\frac{4T_0l^3}{\pi^3}}{\left(\frac{\lambda l^3}{3\rho c_V}+\frac{pl^5}{30}\right)}. \tag{12}$$

Substituting Eq. (12) into Eq. (9), we get

$$F_1^* = \frac{-\frac{4T_0l^3}{\pi^3}}{\left(\frac{\lambda l^3}{3\rho c_V}+\frac{pl^5}{30}\right)}(x^2-lx) + \frac{T_0}{p}. \tag{13}$$

From the inverse Laplace transform of Eq. (13), the approximate solution is obtained:

$$T_1^*(x,t) = -\frac{120}{\pi^3}T_0e^{-10\frac{\lambda t}{\rho c_V l^2}}\left(\frac{x^2}{l^2}-\frac{x}{l}\right) + T_0. \tag{14}$$

The calculation error is

$$\Delta T_1(x,t) = T_1^*(x,t) - T_1(x,t) = -\frac{120}{\pi^3}T_0e^{-10\frac{\lambda t}{\rho c_V l^2}}\left(\frac{x^2}{l^2}-\frac{x}{l}\right) - Te^{-\frac{\lambda\pi^2 t}{\rho c_V l^2}}\sin\frac{\pi x}{l}. \tag{15}$$

From the differentiations of Eq. (15), we have $\frac{\partial(\Delta T_1(x,t))}{\partial t} \leq 0$, $\left.\frac{\partial(\Delta T_1(x,t))}{\partial x}\right|_{0<x\leq\frac{l}{2}} \geq 0$, $\left.\frac{\partial(\Delta T_1(x,t))}{\partial x}\right|_{x=\frac{l}{2}} = 0$ and $\left.\frac{\partial(\Delta T_1(x,t))}{\partial x}\right|_{\frac{l}{2}\leq x<l} \leq 0$. Therefore, the calculation error reaches a maximum at $x = \frac{l}{2}$ and $t = 0$

$$\max[\Delta T_1(x,t)] = \Delta T_1\left(\frac{l}{2}, 0\right) = 0.03245 T_0, \tag{16}$$

which is two orders of magnitude smaller than the equilibrium temperature T_0. For a fixed time t_0, $\Delta T_1\left(\frac{l}{2}, t_0\right)$ is the largest calculation error in the whole temperature field. Therefore, the calculation error at $x = \frac{l}{2}$ could show the degree of approximation in this problem, which is shown by Figure 1. It is found that $\Delta T_1\left(\frac{l}{2}, t\right)$ decays with time. The largest calculation error in the whole field will be smaller than $0.01T_0$ when $Fo = 0.2$, and almost zero when $Fo = 0.8$. Notice that the whole temperature field $T(x, t) \geq T_0$, and therefore the relative error will be smaller than the numerical value in Figure 1. In engineering, when $Fo > 0.2$, the one-term approximate solutions or the transient temperature charts will be applicable because their relative errors are smaller than 1%. It means that for this example, this approximate method can provide sufficient accuracy in engineering.

2.2. CV model

The heat conduction equation of the CV model is

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} = \frac{\lambda}{\rho c_V} \nabla^2 T. \tag{17}$$

The variational principle for the first type of boundary condition [43] is

$$\delta\left\{\iiint_D \left\{\frac{\lambda}{\rho c_V}|\nabla F|^2 + \left(p + \tau p^2\right)F^2 - 2\left[(\tau p + 1)T|_{t=0} + \tau \left.\frac{\partial T}{\partial t}\right|_{t=0}\right]F\right\}dV\right\} = 0. \tag{18}$$

Consider a one-dimensional problem in $0 \leq x \leq l$, where the initial conditions are taken as $T|_{t=0} = T_0\left(1 + \sin\frac{\pi x}{l}\right)$, $\left.\frac{\partial T}{\partial t}\right|_{t=0} = -\frac{T_0}{2\tau}\sin\frac{\pi x}{l}$; the boundary conditions are taken as $T|_{x=0,\ l} = T_0$; and the physical properties satisfy $\frac{4\lambda\tau\pi^2}{\rho c_V l^2} = 1$. The classical solution of this problem is

$$T_{CV}(x,t) = T_0\left(1 + e^{-\frac{t}{2\tau}}\sin\frac{\pi x}{l}\right). \tag{19}$$

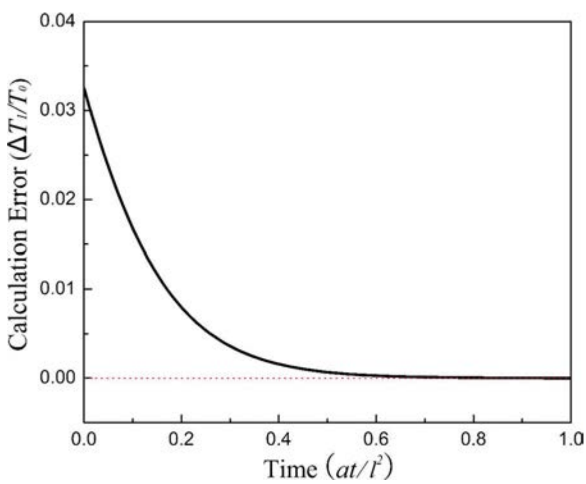

Figure 1. The calculation error of Fourier's law.

The approximate solution of this problem is $T^*_{CV}(x,t)$, whose Laplace transform is F^*_{CV}. Similar to Fourier's law, F^*_{CV} is still written as a second-order polynomial approximation:

$$F^*_{CV} = k_{CV}x^2 + b_{CV}x + c_{CV}. \tag{20}$$

The Laplace transforms of the boundary conditions are

$$F^*_{CV}\big|_{x=0,l} = \frac{T_0}{p}. \tag{21}$$

From Eqs. (20) and (21), we can obtain

$$c_{CV} = \frac{T_0}{p}, \tag{22-a}$$

$$k_{CV}l + b_{CV} = 0. \tag{22-b}$$

Then, F^*_{CV} can be expressed by k_{CV}

$$F^*_{CV} = k_{CV}x^2 - k_{CV}lx + \frac{T_0}{p}. \tag{23}$$

Substituting Eq. (23) into Eq. (18) leads to

$$\delta\left\{\int_0^l \left[\frac{\lambda}{\rho c_V}(2k_{CV}x - k_{CV}l)^2 + (p+\tau p^2)\left(k_{CV}x^2 - k_{CV}lx + \frac{T_0}{p}\right)^2 - 2T_0\left(k_{CV}x^2 - k_{CV}lx + \frac{T_0}{p}\right)\left(1 + \tau p + \tau p\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{\pi x}{l}\right)\right]dx\right\} = 0. \tag{24}$$

Equation (24) can be simplified to

$$\delta\left[\left(\frac{\lambda l^3}{3\rho c_V} + \frac{(p+\tau p^2)l^5}{30}\right)k^2_{CV} + \frac{4T_0l^3k_{CV}}{\pi^3}(1+2\tau p)\right] = 0, \tag{25-a}$$

$$\left[2\left(\frac{\lambda l^3}{3\rho c_V} + \frac{(p+\tau p^2)l^5}{30}\right)k_{CV} + \frac{4T_0l^3}{\pi^3}(1+2\tau p)\right]\delta(k_{CV}) = 0. \tag{25-b}$$

From Eq. (25-b),

$$k_{CV} = \frac{-\frac{2T_0l^3}{\pi^3}(1+2\tau p)}{\frac{\lambda l^3}{3\rho c_V} + \frac{(p+\tau p^2)l^5}{30}}. \tag{26}$$

Substituting Eq. (26) into Eq. (23), we get

$$F^*_{CV} = \frac{-\frac{2T_0l^3}{\pi^3}(1+2\tau p)}{\frac{\lambda l^3}{3\rho c_V} + \frac{(p+\tau p^2)l^5}{30}}\left(x^2 - lx\right) + \frac{T_0}{p}. \tag{27}$$

From $\frac{4\lambda\tau\pi^2}{\rho c_V l^2} = 1$ and the inverse Laplace transform of Eq. (27), the approximate solution is obtained

$$T^*_{CV}(x,t) = -\frac{120}{\pi^3}T_0e^{-\frac{t}{2\tau}}\cos\left(\frac{t}{\tau}\sqrt{\frac{5}{2\pi^2} - \frac{1}{4}}\right)\left(\frac{x^2}{l^2} - \frac{x}{l}\right) + T_0. \tag{28}$$

The calculation error is

$$\begin{aligned}\Delta T_{CV}(x,t) &= T^*_{CV}(x,t) - T_{CV}(x,t)\\ &= -\frac{120}{\pi^3}T_0 e^{-\frac{t}{2\tau}}\cos\left(\frac{t}{\tau}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}}\right)\left(\frac{x^2}{l^2}-\frac{x}{l}\right) - T_0 e^{-\frac{t}{2\tau}}\sin\frac{\pi x}{l}\cdot\end{aligned} \tag{29}$$

From the differentiations of Eq. (29), we have $\frac{\partial(\Delta T_{CV}(x,t))}{\partial t} \le 0$, $\left.\frac{\partial(\Delta T_{CV}(x,t))}{\partial x}\right|_{0<x\le\frac{l}{2}} \ge 0$, $\left.\frac{\partial(\Delta T_{CV}(x,t))}{\partial x}\right|_{x=\frac{l}{2}} = 0$ and $\left.\frac{\partial(\Delta T_{CV}(x,t))}{\partial x}\right|_{\frac{l}{2}\le x<l} \le 0$. Therefore, the calculation error reaches a maximum at $x = \frac{l}{2}$ and $t = 0$

$$\max[\Delta T_{CV}(x,t)] = \Delta T_{CV}\left(\frac{l}{2},0\right) = 0.03245T_0. \tag{30}$$

which is also two orders of magnitude smaller than the equilibrium temperature T_0. For a fixed time t_0, $\Delta T_{CV}\left(\frac{l}{2},t_0\right)$ is still the largest calculation error in the whole field. Figure 2 shows the calculation error at $x = \frac{l}{2}$, which reflects the degree of approximation in this problem. It is found that $\Delta T_{CV}\left(\frac{l}{2},t\right)$ decays with time, which will be smaller than $0.01T_0$ when $t = 2.5\tau$, and almost zero when $t = 10\tau$. In general, the relaxation time of matters is very small in physics, which is in the order of *ps~fs*, showing that the calculation error will tend to zero very quickly.

2.3. Jeffrey model

The heat conduction equation of the Jeffrey model is

$$\frac{1}{\tau}\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial t^2} = \frac{k}{\rho c_V \tau}\nabla^2 T + \frac{k_F}{\rho c_V}\frac{\partial}{\partial t}\left(\nabla^2 T\right). \tag{31}$$

The variational principle for the first type of boundary condition [40] is

$$\delta\left\{\iiint_D \left\{\begin{array}{l}\left(\frac{k}{\rho c_V \tau}+\frac{pk_F}{\rho c_V}\right)|\nabla F|^2 + \left(p^2+\frac{p}{\tau}\right)F^2 - \\ 2\left[\left(p+\frac{1}{\tau}\right)T\big|_{t=0} + \frac{\partial T}{\partial t}\Big|_{t=0} - \frac{k_F}{\rho c_V}\nabla^2 T\big|_{t=0}\right]F\end{array}\right\} dV\right\} = 0. \tag{32}$$

Consider a one-dimensional problem in $0 \le x \le l$, where the initial conditions are taken as $T|_{t=0} = T_0\left(1+\sin\frac{\pi x}{l}\right)$, $\left.\frac{\partial T}{\partial t}\right|_{t=0} = -\frac{T_0}{2\tau}\sin\frac{\pi x}{l}$, $\nabla^2 T|_{t=0} = -\frac{\pi^2}{l^2}T_0\sin\frac{\pi x}{l}$; the boundary conditions are

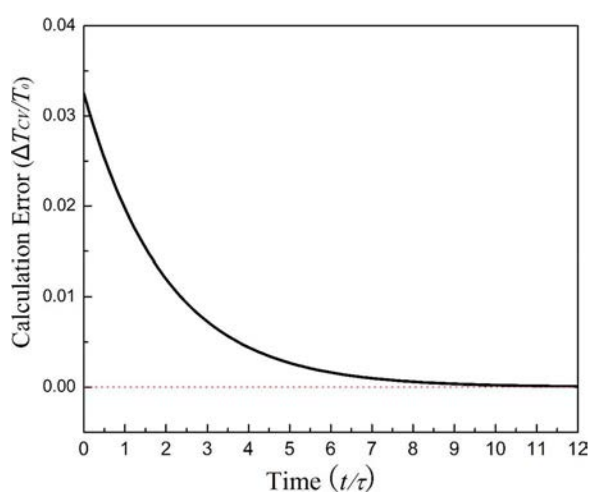

Figure 2. The calculation error of the CV model.

taken as $T|_{x=0,\,l} = T_0$; and the physical properties satisfy $\frac{4k\tau\pi^2}{\rho c_V l^2} = \left(1 + \frac{k_F\tau\pi^2}{\rho c_V l^2}\right)^2$. The classical solution of this problem is

$$T(x,t) = T_0\left(1 + e^{-\frac{t}{2\tau} - \frac{k_F\pi^2 t}{2\rho c_V l^2}} \sin\frac{\pi x}{l}\right). \tag{33}$$

Similar to the above-mentioned cases, F_J^*, which is the Laplace transform of the approximate solution $T_J^*(x,t)$, is still expressed as a second-order polynomial approximation:

$$F_J^* = k_J x^2 - k_J l x + \frac{T_0}{p}. \tag{34}$$

Substituting Eq. (34) into Eq. (32) leads to

$$\delta\left\{\int_0^l \left[\left(\frac{k}{\rho c_V} + \frac{\tau p k_F}{\rho c_V}\right)(2k_J x - k_J l)^2 + \left(p + \tau p^2\right)\left(k_J x^2 - k_J l x + \frac{T_0}{p}\right)^2 - 2T_0 \right.\right.$$
$$\left.\left.\left(k_J x^2 - k_J l x + \frac{T_0}{p}\right)\left(1 + \tau p + \tau p \sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{\pi x}{l} + \frac{\tau k_F \pi^2}{\rho c_V l^2}\sin\frac{\pi x}{l}\right)\right] dx\right\} = 0. \tag{35}$$

Equation (35) could be simplified to

$$\delta\left\{\left[\frac{(k+\tau p k_F) l^3}{3\rho c_V} + \frac{(p+\tau p^2) l^5}{30}\right] k_J^2 + \frac{4T_0 l^3 k_J}{\pi^3}\left(1 + 2\tau p + \frac{2\tau k_F \pi^2}{\rho c_V l^2}\right)\right\} = 0, \tag{36-a}$$

$$\left\{2\left[\frac{(k+\tau p k_F) l^3}{3\rho c_V} + \frac{(p+\tau p^2) l^5}{30}\right] k_J + \frac{4T_0 l^3}{\pi^3}\left(1 + 2\tau p + \frac{2\tau k_F \pi^2}{\rho c_V l^2}\right)\right\}\delta(k_J) = 0. \tag{36-b}$$

From Eq. (36-b),

$$k_J = \frac{-\frac{2T_0 l^3}{\pi^3}\left(1 + 2\tau p + \frac{2\tau k_F\pi^2}{\rho c_V l^2}\right)}{\frac{(k+\tau p k_F) l^3}{3\rho c_V} + \frac{(p+\tau p^2) l^5}{30}}. \tag{37}$$

Then, we can obtain F_J^*

$$F_J^* = \frac{-\frac{2T_0 l^3}{\pi^3}\left(1 + 2\tau p + \frac{2\tau k_F\pi^2}{\rho c_V l^2}\right)}{\frac{(k+\tau p k_F) l^3}{3\rho c_V} + \frac{(p+\tau p^2) l^5}{30}}\left(x^2 - l x\right) + \frac{T_0}{p}. \tag{38}$$

There are three cases for the inverse Laplace transform of Eq. (38). When $\frac{1}{\tau^2}\left(\frac{5}{2\pi^2} - \frac{1}{4}\right) + \frac{k_F^2}{\rho^2 c_V^2 l^4}\left(\frac{5\pi^2}{2} - 25\right) > 0$ (called **Jeffrey-1**), the approximate solution is

$$T_J^*(x,t) = T_0 - \frac{120T_0}{\pi^3}\left(\frac{x^2}{l^2} - \frac{x}{l}\right)e^{-\frac{t}{2\tau} - \frac{5k_F t}{\rho c_V l^2}}\left\{\cos\left[\frac{t}{\tau}\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{k_F^2\tau^2}{\rho^2 c_V^2 l^4}\left(\frac{5\pi^2}{2} - 25\right)}\right]\right.$$
$$\left. + \frac{k_F\tau(\pi^2 - 5)}{\rho c_V l^2\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{k_F^2\tau^2}{\rho^2 c_V^2 l^4}\left(\frac{5\pi^2}{2} - 25\right)}}\sin\left[\frac{t}{\tau}\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{k_F^2\tau^2}{\rho^2 c_V^2 l^4}\left(\frac{5\pi^2}{2} - 25\right)}\right]\right\}. \tag{39}$$

The calculation error is

$$\Delta T_{J1}(x,t) = -\frac{120T_0}{\pi^3}\left(\frac{x^2}{l^2}-\frac{x}{l}\right)e^{-\frac{t}{2\tau}-\frac{5k_Ft}{\rho c_V l^2}}\left\{\cos\left[\frac{t}{\tau}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(\frac{5\pi^2}{2}-25\right)}\right]\right.$$
$$\left.+\frac{k_F\tau(\pi^2-5)}{\rho c_V l^2\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(\frac{5\pi^2}{2}-25\right)}}\sin\left[\frac{t}{\tau}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(\frac{5\pi^2}{2}-25\right)}\right]\right\} \quad (40)$$
$$-T_0e^{-\frac{t}{2\tau}-\frac{k_F\pi^2t}{2\rho c_V l^2}}\sin\frac{\pi x}{l}.$$

In this case, the calculation error will still reach a maximum at $x=\frac{l}{2}$ and $t=0$, $\max[\Delta T_{J1}(x,t)] = \Delta T_{J1}\left(\frac{l}{2},0\right) = 0.03245T_0$, which is still two orders of magnitude smaller than the equilibrium temperature T_0. $\Delta T_{J1}\left(\frac{l}{2},t\right)$ is shown in Figure 3 (**Jeffrey-1**, $\frac{k_F\tau}{\rho c_V l^2}=\frac{1}{2\pi^2}$). In this case, the largest calculation error in the whole temperature field still decays with time, which will be smaller than $0.01T_0$ when $t=2\tau$, and almost zero when $t=9\tau$. When $\frac{1}{\tau^2}\left(\frac{5}{2\pi^2}-\frac{1}{4}\right)+\frac{k_F^2}{\rho^2c_V^2l^4}\left(\frac{5\pi^2}{2}-25\right)=0$ (called **Jeffrey-2**), the approximate solution is

$$T_J^*(x,t) = T_0 - \frac{120T_0}{\pi^3}\left(\frac{x^2}{l^2}-\frac{x}{l}\right)e^{-\frac{t}{2\tau}-\frac{5k_Ft}{\rho c_V l^2}}\left[1+\frac{k_F(\pi^2-5)}{\rho c_V l^2}t\right]. \quad (41)$$

The calculation error is

$$\Delta T_{J2}(x,t) = \frac{120T_0}{\pi^3}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)e^{-\frac{t}{2\tau}-\frac{5k_Ft}{\rho c_V l^2}}\left[1+\frac{k_F(\pi^2-5)}{\rho c_V l^2}t\right]-T_0e^{-\frac{t}{2\tau}-\frac{k_F\pi^2t}{2\rho c_V l^2}}\sin\frac{\pi x}{l}. \quad (42)$$

In this case, $\Delta T_{J2}\left(\frac{l}{2},t\right)$, shown in Figure 3 (**Jeffrey-2**), is still the largest calculation error in the whole temperature field, but $\Delta T_{J2}\left(\frac{l}{2},0\right)$ is not the maximum error. The largest calculation error in the whole field reaches its maximum at $t\approx\tau$, and this maximum error is only one order of magnitude smaller than the equilibrium temperature T_0. When $\frac{1}{\tau^2}\left(\frac{5}{2\pi^2}-\frac{1}{4}\right)+\frac{k_F^2}{\rho^2c_V^2l^4}\left(\frac{5\pi^2}{2}-25\right)<0$ (called **Jeffrey-3**), the

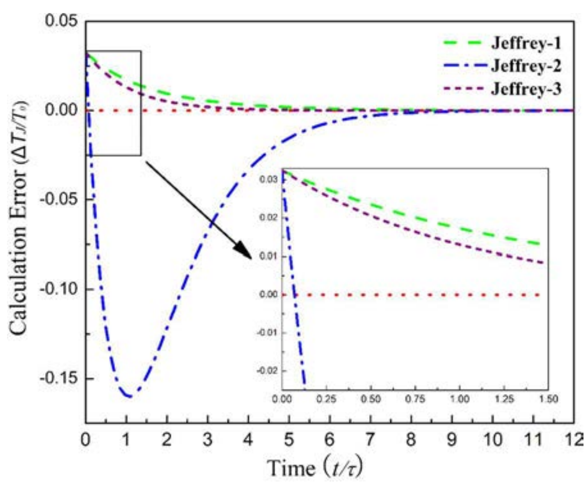

Figure 3. The calculation error of the Jeffrey model.

approximate solution is

$$T_J^*(x,t) = -\frac{60T_0}{\pi^3}\left(\frac{x^2}{l^2}-\frac{x}{l}\right)e^{-\frac{t}{2\tau}-\frac{5k_Ft}{\rho c_V l^2}-\frac{t}{\tau}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}\left\{e^{\frac{2t}{\tau}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}+1\right.$$
$$\left.+\frac{k_F\tau(\pi^2-5)}{2\rho c_V l^2\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}\left[e^{\frac{2t}{\tau}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}-1\right]\right\}+T_0. \tag{43}$$

The calculation error is

$$\Delta T_{J3}(x,t) = -\frac{60T_0}{\pi^3}\left(\frac{x^2}{l^2}-\frac{x}{l}\right)e^{-\frac{t}{2\tau}-\frac{5k_Ft}{\rho c_V l^2}-\frac{t}{\tau}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}$$
$$\left\{e^{\frac{2t}{\tau}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}+1+\frac{k_F\tau(\pi^2-5)}{2\rho c_V l^2\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}\right.$$
$$\left.\left[e^{\frac{2t}{\tau}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{k_F^2\tau^2}{\rho^2c_V^2l^4}\left(25-\frac{5\pi^2}{2}\right)}}-1\right]\right\}-T_0e^{-\frac{t}{2\tau}-\frac{k_F\pi^2t}{2\rho c_V l^2}}\sin\frac{\pi x}{l}. \tag{44}$$

In this case, the calculation error will still reach the maximum at $x=\frac{l}{2}$ and $t=0$, $\max[\Delta T_{J3}(x,t)] = \Delta T_{J3}\left(\frac{l}{2},0\right) = 0.03245T_0$, which is still two orders of magnitude smaller than the equilibrium temperature T_0. Figure 3 $\left(\textbf{Jeffrey} - \mathbf{3}\,\frac{k_F\tau}{\rho c_V l^2}=\frac{1}{\pi^2}\right)$ shows the largest calculation error in the whole field. $\Delta T_{J3}\left(\frac{l}{2},t\right)$, which decays with time, will be smaller than $0.01T_0$ when $t=1.5\tau$, and almost zero when $t=5.5\tau$.

2.4. TT model

Anisinov et al. proposed the TT model for metals with the heat conduction equation expressed as

$$\nabla^2T_e+\frac{\alpha_e}{C_E^2}\frac{\partial}{\partial t}\left(\nabla^2T_e\right)=\frac{1}{\alpha_E}\frac{\partial T_e}{\partial t}+\frac{1}{C_E^2}\frac{\partial^2T_e}{\partial t^2}, \tag{45}$$

For the first type of boundary condition, the variational principle [40] is

$$\delta\left\{\iiint_D\left\{\left(1+\frac{p\alpha_e}{C_E^2}\right)|\nabla F_e|^2+\left(\frac{p}{\alpha_E}+\frac{p^2}{C_E^2}\right)F_e^2-2\right.\right.$$
$$\left.\left.\left[\frac{1}{C_E^2}\left(p\nabla T_e|_{t=0}+\frac{\partial T_e}{\partial t}\bigg|_{t=0}\right)+\frac{1}{\alpha_E}\nabla T_e|_{t=0}-\frac{\alpha_e}{C_E^2}\nabla^2T_e|_{t=0}\right]F_e\right\}dV\right\}=0. \tag{46}$$

Consider a one-dimensional problem in $0\le x\le l$, where the initial conditions are taken as $T|_{t=0}=T_0\left(1+\sin\frac{\pi x}{l}\right)$, $\frac{\partial T}{\partial t}\big|_{t=0}=-\frac{T_0C_E^2}{2\alpha_E}\sin\frac{\pi x}{l}$, $\nabla^2T|_{t=0}=-\frac{\pi^2}{l^2}T_0\sin\frac{\pi x}{l}$; the boundary conditions are taken as $T|_{x=0,\,l}=T_0$; and the physical properties satisfy $\frac{4\alpha_E^2\pi^2}{l^2C_E^2}=\left(1+\frac{\alpha_e\alpha_E\pi^2}{C_E^2l^2}\right)^2$. The classical solution of this problem is

$$T(x,t)=T_0\left(1+e^{-\frac{tC_E^2}{2\alpha_E}-\frac{\alpha_e\pi^2t}{2l^2}}\sin\frac{\pi x}{l}\right). \tag{47}$$

F_e^*, which is the Laplace transform of the approximate solution $T_e^*(x,t)$, can be written as

$$F_e^* = k_e x^2 - k_e l x + \frac{T_0}{p}. \tag{48}$$

Substituting Eq. (48) into Eq. (46) leads to

$$\delta\left\{\int_0^l\left[\left(\alpha_E + \frac{p\alpha_e\alpha_E}{C_E^2}\right)(2k_e x - k_e l)^2 + \left(p + \frac{p^2\alpha_E}{C_E^2}\right)\left(k_e x^2 - k_e l x + \frac{T_0}{p}\right)^2 - .2T_0\right.\right.$$
$$\left.\left.\left(k_e x^2 - k_e l x + \frac{T_0}{p}\right)\left[\left(1 + \frac{p\alpha_E}{C_E^2}\right)\left(1 + \sin\frac{\pi x}{l}\right) - \frac{1}{2}\sin\frac{\pi x}{l} + \frac{\alpha_e\alpha_E}{C_E^2}\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}\right]dx\right\} = 0. \tag{49}$$

Equation (49) could be simplified to

$$\left\{2\left[\frac{\left(\alpha_E + \frac{p\alpha_e\alpha_E}{C_E^2}\right)l^3}{3} + \frac{\left(p + \frac{p^2\alpha_E}{C_E^2}\right)l^5}{30}\right]k_e + \frac{4T_0 l^3}{\pi^3}\left(1 + \frac{2p\alpha_E}{C_E^2} + \frac{2\pi^2\alpha_e\alpha_E}{l^2 C_E^2}\right)\right\}\delta(k_e) = 0. \tag{50}$$

From Eq. (50),

$$k_e = \frac{-\frac{2T_0 l^3}{\pi^3}\left(1 + \frac{2p\alpha_E}{C_E^2} + \frac{2\pi^2\alpha_e\alpha_E}{l^2 C_E^2}\right)}{\frac{\left(\alpha_E + \frac{p\alpha_e\alpha_E}{C_E^2}\right)l^3}{3} + \frac{\left(p + \frac{p^2\alpha_E}{C_E^2}\right)l^5}{30}}. \tag{51}$$

Then, we can obtain the Laplace transform of the approximate solution

$$F_e^* = \frac{-\frac{2T_0 l^3}{\pi^3}\left(1 + \frac{2p\alpha_E}{C_E^2} + \frac{2\pi^2\alpha_e\alpha_E}{l^2 C_E^2}\right)}{\frac{\left(\alpha_E + \frac{p\alpha_e\alpha_E}{C_E^2}\right)l^3}{3} + \frac{\left(p + \frac{p^2\alpha_E}{C_E^2}\right)l^5}{30}}\left(x^2 - lx\right) + \frac{T_0}{p}. \tag{52}$$

There are also three cases for the inverse Laplace transform of Eq. (52). When $\frac{C_E^4}{\alpha_E^2}\left(\frac{5}{2\pi^2} - \frac{1}{4}\right) + \frac{\alpha_e^2}{l^4}\left(\frac{5\pi^2}{2} - 25\right) > 0$ (called **TT-1**), the approximate solution is

$$T_e^*(x,t) = -\frac{120T_0}{\pi^3}\left(\frac{x^2}{l^2} - \frac{x}{l}\right)e^{-\frac{C_E^2 t}{2\alpha_E} - \frac{5\alpha_e t}{l^2}}$$
$$\left\{\cos\left[\frac{tC_E^2}{\alpha_E}\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(\frac{5\pi^2}{2} - 25\right)}\right] + \frac{\alpha_E\alpha_e(\pi^2 - 5)}{l^2 C_E^2\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{a_F^2\tau^2}{l^4}\left(\frac{5\pi^2}{2} - 25\right)}}\right.$$
$$\left.\sin\left[\frac{tC_E^2}{\alpha_E}\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(\frac{5\pi^2}{2} - 25\right)}\right]\right\} + T_0. \tag{53}$$

The calculation error is

$$\Delta T_{e1}(x,t) = \frac{120T_0}{\pi^3}\left(\frac{x}{l} - \frac{x^2}{l^2}\right)e^{-\frac{C_E^2 t}{2\alpha_E} - \frac{5\alpha_e t}{l^2}}$$
$$\left\{\cos\left[\frac{tC_E^2}{\alpha_E}\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(\frac{5\pi^2}{2} - 25\right)}\right] + \frac{\alpha_E\alpha_e(\pi^2 - 5)}{l^2 C_E^2\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{a_F^2\tau^2}{l^4}\left(\frac{5\pi^2}{2} - 25\right)}}\right.$$
$$\left.\sin\left[\frac{tC_E^2}{\alpha_E}\sqrt{\frac{5}{2\pi^2} - \frac{1}{4} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(\frac{5\pi^2}{2} - 25\right)}\right]\right\} - T_0 e^{-\frac{tC_E^2}{2\alpha_E} - \frac{\alpha_e\pi^2 t}{2l^2}}\sin\frac{\pi x}{l}. \tag{54}$$

In this case, the calculation error will still reach the maximum at $x = \frac{l}{2}$ and $t = 0$, $\max[\Delta T_{e1}(x,t)] = \Delta T_{e1}\left(\frac{l}{2}, 0\right) = 0.03245T_0$, which is still two orders of magnitude smaller than the equilibrium temperature T_0. Figure 4 (**TT-1**, $\frac{\alpha_e \alpha_E}{l^2 C_E^2} = \frac{1}{2\pi^2}$, $\tau_E = \frac{\alpha_E}{C_E^2}$) shows $\Delta T_{e1}\left(\frac{l}{2}, 0\right)$, which will be smaller than $0.01T_0$ when $t = 2\tau_E$, and almost zero when $t = 9\tau_E$. When $\frac{C_E^4}{\alpha_E^2}\left(\frac{5}{2\pi^2} - \frac{1}{4}\right) + \frac{\alpha_e^2}{l^4}\left(\frac{5\pi^2}{2} - 25\right) = 0$ (called **TT-2**), the approximate solution is

$$T_e^*(x,t) = T_0 - \frac{120T_0}{\pi^3}\left(\frac{x^2}{l^2} - \frac{x}{l}\right)e^{-\frac{C_E^2 t}{2\alpha_E} - \frac{5\alpha_e t}{l^2}}\left[1 + \frac{\alpha_e(\pi^2 - 5)}{l^2}t\right]. \tag{55}$$

The calculation error is

$$\begin{aligned}\Delta T_{e2}(x,t) = {}& \frac{120T_0}{\pi^3}\left(\frac{x}{l} - \frac{x^2}{l^2}\right)e^{-\frac{C_E^2 t}{2\alpha_E} - \frac{5\alpha_e t}{l^2}}\left[1 + \frac{\alpha_e(\pi^2 - 5)}{l^2}t\right] \\ & - T_0 e^{-\frac{tC_E^2}{2\alpha_E} - \frac{\alpha_e \pi^2 t}{2l^2}} \sin\frac{\pi x}{l}.\end{aligned} \tag{56}$$

$\Delta T_{e2}\left(\frac{l}{2}, t\right)$, shown in Figure 4 (**TT-2**), is still the largest calculation error in the whole temperature field. The largest calculation error in the whole field reaches its maximum at $t \approx \tau_E$, which is only one order of magnitude smaller than the equilibrium temperature T_0. When $\frac{C_E^4}{\alpha_E^2}\left(\frac{5}{2\pi^2} - \frac{1}{4}\right) + \frac{\alpha_e^2}{l^4}\left(\frac{5\pi^2}{2} - 25\right) < 0$ (called **TT-3**), the approximate solution is

$$\begin{aligned}T_e^*(x,t) = {}& -\frac{60T_0}{\pi^3}\left(\frac{x^2}{l^2} - \frac{x}{l}\right)e^{-\frac{C_E^2 t}{2\alpha_E} - \frac{5\alpha_e t}{l^2} - \frac{tC_E^2}{\alpha_E}\sqrt{\frac{1}{4} - \frac{5}{2\pi^2} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25 - \frac{5\pi^2}{2}\right)}}\left\{e^{\frac{2tC_E^2}{\alpha_E}\sqrt{\frac{1}{4} - \frac{5}{2\pi^2} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25 - \frac{5\pi^2}{2}\right)}} + 1\right. \\ & \left. + \frac{\alpha_E\alpha_e(\pi^2 - 5)}{2l^2 C_E^2\sqrt{\frac{1}{4} - \frac{5}{2\pi^2} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25 - \frac{5\pi^2}{2}\right)}}\left[e^{\frac{2tC_E^2}{\alpha_E}\sqrt{\frac{1}{4} - \frac{5}{2\pi^2} + \frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25 - \frac{5\pi^2}{2}\right)}} - 1\right]\right\} + T_0.\end{aligned} \tag{57}$$

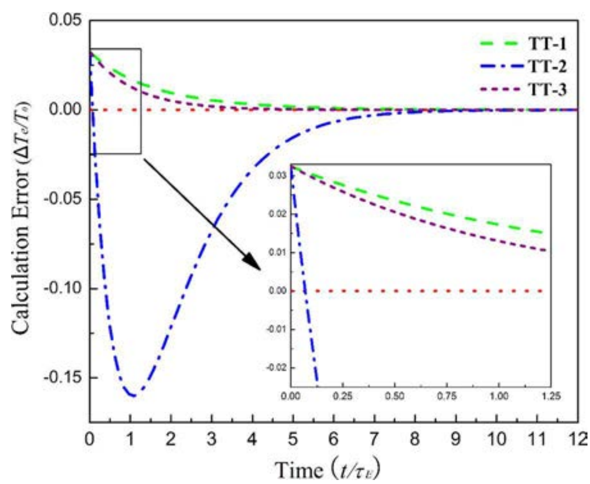

Figure 4. The calculation error of the TT model.

The calculation error is

$$\Delta T_{e3}(x,t)=\frac{60T_0}{\pi^3}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)e^{-\frac{C_E^2 t}{2\alpha_E}-\frac{5\alpha_e t}{l^2}-\frac{tC_E^2}{\alpha_E}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25-\frac{5\pi^2}{2}\right)}}\left\{e^{\frac{2tC_E^2}{\alpha_E}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25-\frac{5\pi^2}{2}\right)}}+1\right.$$
$$\left.+\frac{\alpha_E\alpha_e(\pi^2-5)}{2l^2C_E^2\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25-\frac{5\pi^2}{2}\right)}}\left[e^{\frac{2tC_E^2}{\alpha_E}\sqrt{\frac{1}{4}-\frac{5}{2\pi^2}+\frac{\alpha_e^2\alpha_E^2}{C_E^4 l^4}\left(25-\frac{5\pi^2}{2}\right)}}-1\right]\right\}-T_0e^{-\frac{tC_E^2}{2\alpha_E}-\frac{\alpha_e\pi^2 t}{2l^2}}\sin\frac{\pi x}{l}. \tag{58}$$

In this case, the calculation error will still reach the maximum at $x=\frac{l}{2}$ and $t=0$, $\max[\Delta T_{e3}(x,t)]=\Delta T_{e3}\left(\frac{l}{2},0\right)=0.03245T_0$, which is still two orders of magnitude smaller than the equilibrium temperature T_0. Figure 4 (**TT-3**, $\frac{\alpha_e\alpha_E}{l^2C_E^2}=\frac{1}{\pi^2}$) shows $\Delta T_{e3}\left(\frac{l}{2},t\right)$, which will be smaller than $0.01T_0$ when $t=1.5\tau_E$, and almost zero when $t=5.5\tau_E$.

2.5. GK model

The GK model is a classical model for phonon heat conduction whose heat conduction equation is

$$\nabla^2 T+\frac{9\tau_N}{5}\frac{\partial}{\partial t}(\nabla^2 T)=\frac{2}{\tau_R c^2}\frac{\partial T}{\partial t}+\frac{3}{c^2}\frac{\partial^2 T}{\partial t^2}, \tag{59}$$

For the first type of boundary condition, the variational principle [40] is

$$\delta\left\{\iiint_D\left\{\begin{array}{l}\left(1+\frac{9\tau_N}{5}\right)|\nabla F_e|^2+\left(\frac{2p}{\tau_R c^2}+\frac{3p^2}{c^2}\right)F^2- \\ 2\left[\left(\frac{2}{\tau_R c^2}+\frac{3p}{c^2}\right)T\Big|_{t=0}+\frac{3}{c^2}\frac{\partial T}{\partial t}\Big|_{t=0}-\frac{9\tau_N}{5}\nabla^2 T|_{t=0}\right]F\end{array}\right\}dV\right\}=0. \tag{60}$$

Consider a one-dimensional problem in $0\le x\le l$, where the initial conditions are taken as $T|_{t=0}=T_0\left(1+\sin\frac{\pi x}{l}\right)$, $\frac{\partial T}{\partial t}\Big|_{t=0}=-\frac{T_0}{3\tau_R}\sin\frac{\pi x}{l}$, $\nabla^2 T|_{t=0}=-\frac{\pi^2}{l^2}T_0\sin\frac{\pi x}{l}$; the boundary conditions are taken as $T|_{x=0,\ l}=T_0$; and the physical properties satisfy $\frac{3\tau_R^2c^2\pi^2}{l^2}=\left(1+\frac{9\tau_N\tau_R c^2}{10}\frac{\pi^2}{l^2}\right)^2$. The classical solution of this problem is

$$T(x,t)=T_0\left(1+e^{-\frac{t}{2\tau}-\frac{3\tau_N c^2}{10}\frac{\pi^2 t}{l^2}}\sin\frac{\pi x}{l}\right). \tag{61}$$

As the Laplace transform of the approximate solution $T_{GK}^*(x,t)$, F_{GK}^* is still written as

$$F_{GK}^*=k_{GK}x^2-k_{GK}lx+\frac{T_0}{p}. \tag{62}$$

Substituting Eq. (62) into Eq. (60) leads to

$$\begin{aligned}&\delta\int_0^l\left[\left(\frac{\tau_R c^2}{2}+\frac{9p\tau_N\tau_R c^2}{10}\right)(2k_{GK}x-k_{GK}l)^2+\left(p+\frac{3\tau_R p^2}{2}\right)\right.\\&\left(k_{GK}x^2-k_{GK}lx+\frac{T_0}{p}\right)^2-2T_0\left(k_{GK}x^2-k_{GK}lx+\frac{T_0}{p}\right)\\&\left.\left[\left(1+\frac{3\tau_R p}{2}\right)\left(1+\sin\frac{\pi x}{l}\right)-\frac{1}{2}\sin\frac{\pi x}{l}+\frac{9\tau_N\tau_R c^2}{10}\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}\right]=0.\right.\end{aligned} \tag{63}$$

Equation (63) could be simplified to

$$\left\{\begin{array}{l} 2\left[\dfrac{\left(\frac{\tau_R c^2}{2}+\frac{9\tau_N\tau_R c^2 p}{10}\right)l^3}{3}+\dfrac{\left(p+\frac{3\tau_R p^2}{2}\right)l^5}{30}\right]k_{GK} \\ +\dfrac{4T_0 l^3}{\pi^3}\left(1+3\tau_R p+\dfrac{9\tau_N\tau_R c^2}{5}\dfrac{\pi^2}{l^2}\right)\right\}\delta(k_{GK})=0 \end{array}. \tag{64}$$

From Eq. (64),

$$k_{GK}=\frac{-\frac{2T_0 l^3}{\pi^3}\left(1+3\tau_R p+\frac{9\tau_N\tau_R c^2}{5}\frac{\pi^2}{l^2}\right)}{\frac{\left(\frac{\tau_R c^2}{2}+p\frac{9\tau_N\tau_R c^2}{10}\right)l^3}{3}+\frac{\left(p+\frac{3\tau_R}{2}p^2\right)l^5}{30}}. \tag{65}$$

Then, the Laplace transform of the approximate solution is obtained as

$$F^*_{GK}=\frac{-\frac{2T_0 l^3}{\pi^3}\left(1+3\tau_R p+\frac{9\tau_N\tau_R c^2}{5}\frac{\pi^2}{l^2}\right)}{\frac{\left(\frac{\tau_R c^2}{2}+p\frac{9\tau_N\tau_R c^2}{10}\right)l^3}{3}+\frac{\left(p+\frac{3\tau_R}{2}p^2\right)l^5}{30}}\left(x^2-lx\right)+\frac{T_0}{p}. \tag{66}$$

There are also three cases for the inverse Laplace transform of Eq. (66). When $\frac{4}{9\tau_R^2}\left(\frac{5}{2\pi^2}-\frac{1}{4}\right)+\frac{9\tau_N^2 c^4}{25l^4}\left(\frac{5\pi^2}{2}-25\right)>0$ (called **GK-1**), the approximate solution is

$$\begin{aligned} T^*_{GK}(x,t)=&-\frac{120T_0}{\pi^3}\left(\frac{x^2}{l^2}-\frac{x}{l}\right)e^{-\frac{t}{3\tau_R}-\frac{3\tau_N c^2 t}{l^2}}\left\{\cos\left[\frac{2t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2 c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}\right]\right. \\ &\left.+\frac{9\tau_N\tau_R c^2(\pi^2-5)}{10l^2\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}\sin\left[\frac{2t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2 c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}\right]\right\}+T_0. \end{aligned} \tag{67}$$

The calculation error is

$$\begin{aligned} \Delta T_{GK1}(x,t)=&\frac{120T_0}{\pi^3}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)e^{-\frac{t}{3\tau_R}-\frac{3\tau_N c^2 t}{l^2}}\left\{\cos\left[\frac{2t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2 c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}\right]\right. \\ &\left.+\frac{9\tau_N\tau_R c^2(\pi^2-5)}{10l^2\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}\sin\left[\frac{2t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2 c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}\right]\right\} \\ &-T_0 e^{-\frac{t}{2\tau}-\frac{3\tau_N c^2\pi^2 t}{10\,l^2}}\sin\frac{\pi x}{l}. \end{aligned} \tag{68}$$

In this case, the calculation error will still reach the maximum at $x=\frac{l}{2}$ and $t=0$, $\max[\Delta T_{GK1}(x,t)]=\Delta T_{GK1}\left(\frac{l}{2},0\right)=0.03245T_0$, which is still two orders of magnitude smaller than the equilibrium temperature T_0. Figure 5 (**GK-1**, $\frac{9\tau_R\tau_N c^2}{10l^2}=\frac{1}{2\pi^2}$) shows $\Delta T_{GK1}\left(\frac{l}{2},t\right)$, which will be smaller than $0.01T_0$ when $t=3\tau_R$, and almost zero when $t=12\tau_R$. When $\frac{4}{9\tau_R^2}\left(\frac{5}{2\pi^2}-\frac{1}{4}\right)+\frac{9\tau_N^2c^4}{25l^4}\left(\frac{5\pi^2}{2}-25\right)=0$ (called **GK-2**), the approximate solution is

$$T^*_{GK}(x,t)=T_0-\frac{120}{\pi^3}T_0\left(\frac{x^2}{l^2}-\frac{x}{l}\right)e^{-\frac{t}{3\tau_R}-\frac{3\tau_N c^2 t}{l^2}}\left[1-\frac{3\tau_N c^2(\pi^2-5)}{5l^2}t\right]. \tag{69}$$

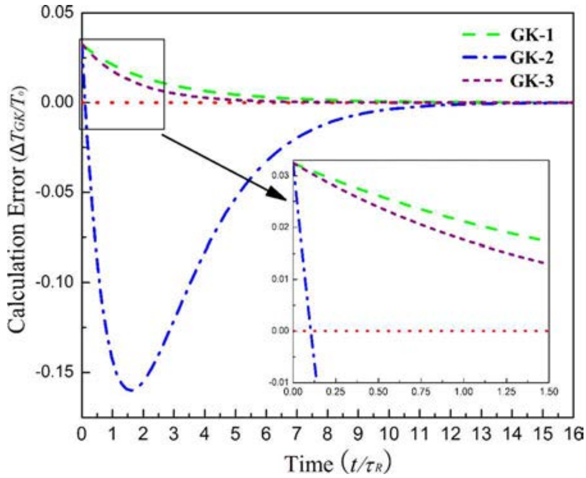

Figure 5. The calculation error of the GK model.

The calculation error is

$$\Delta T_{GK2}(x,t)=\frac{120T_0}{\pi^3}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)e^{-\frac{t}{3\tau_R}-\frac{3\tau_N c^2 t}{l^2}}\left[1-\frac{3\tau_N c^2(\pi^2-5)t}{5l^2}\right]-T_0 e^{-\frac{t}{2\tau}-\frac{3\tau_N c^2}{10}\frac{\pi^2 t}{l^2}}\sin\frac{\pi x}{l}. \tag{70}$$

$\Delta T_{GK2}\left(\frac{l}{2},t\right)$, shown in Figure 5 (**GK-2**), is still the largest calculation error in the whole temperature field. The largest calculation error in the whole field reaches its maximum at $t\approx 1.5\tau_R$, which is only one order of magnitude smaller than the equilibrium temperature T_0. When $\frac{4}{9\tau_R^2}\left(\frac{5}{2\pi^2}-\frac{1}{4}\right)+\frac{9\tau_N^2c^4}{25l^4}\left(\frac{5\pi^2}{2}-25\right)<0$ (called **GK-3**), the approximate solution is

$$\begin{aligned}T_{GK}^*(x,t)=&-\frac{60T_0}{\pi^3}\left(\frac{x^2}{l^2}-\frac{x}{l}\right)e^{-\frac{t}{3\tau_R}-\frac{3\tau_N c^2 t}{l^2}-\frac{2t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}\left\{e^{\frac{4t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}+1\right.\\&\left.+\frac{9\tau_N\tau_R c^2(\pi^2-5)}{20l^2\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}\left[e^{\frac{4t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}-1\right]\right\}+T_0.\end{aligned} \tag{71}$$

The calculation error is

$$\begin{aligned}\Delta T_{GK3}(x,t)=&\frac{60T_0}{\pi^3}\left(\frac{x}{l}-\frac{x^2}{l^2}\right)e^{-\frac{t}{3\tau_R}-\frac{3\tau_N c^2 t}{l^2}-\frac{2t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}\left\{e^{\frac{4t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}+1\right.\\&\left.+\frac{9\tau_N\tau_R c^2(\pi^2-5)}{20l^2\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}\left[e^{\frac{4t}{3\tau_R}\sqrt{\frac{5}{2\pi^2}-\frac{1}{4}+\frac{81\tau_N^2\tau_R^2c^4}{100l^4}\left(\frac{5\pi^2}{2}-25\right)}}-1\right]\right\}\\&-T_0 e^{-\frac{t}{2\tau}-\frac{3\tau_N c^2}{10}\frac{\pi^2 t}{l^2}}\sin\frac{\pi x}{l}.\end{aligned} \tag{72}$$

In this case, the calculation error will still reach a maximum at $x=\frac{l}{2}$ and $t=0$, $\max[\Delta T_{GK3}(x,t)]=\Delta T_{GK3}\left(\frac{l}{2},0\right)=0.03245T_0$, which is still two orders of magnitude smaller than the equilibrium temperature T_0. Figure 5 (**GK-3**, $\frac{9\tau_R\tau_N c^2}{10l^2}=\frac{1}{\pi^2}$) shows $\Delta T_{GK3}\left(\frac{l}{2},0\right)$, which will be smaller than $0.01T_0$ when $t=2\tau_R$, and almost zero when $t=8\tau_R$.

3. Conclusions

This paper applies the variational principles based on the Laplace transforms to the approximate method for both Fourier and non-Fourier heat conduction problems. The approximate analyses can be considered as an extension of the Rayleigh–Ritz variation method. First, in the Laplace transform space, suitable expressions of the trial functions satisfying all boundary conditions are sought. Then, these trial functions are substituted into the variational principles based on Laplace transforms to obtain the undetermined coefficients. After determining the coefficients and trial functions, approximate solutions can be derived from the inverse Laplace transforms of the trial functions. Approximate analytical examples are provided and discussed for one-dimensional problems with the first type of boundary condition and different heat conduction models.

For Fourier's law, the largest calculation error in the whole field decays with time, whose maximum is two orders of magnitude smaller than the equilibrium temperature. The largest calculation error will smaller than $0.01T_0$ when $Fo = 0.2$, which is in the same order of the one-term approximate solutions or the transient temperature charts. This shows that the approximate method based on the variational principles in Ref [40] can provide sufficient accuracy in engineering.

For the CV model, the largest calculation error in the whole field still decays with time, and its maximum is also two orders of magnitude smaller than the equilibrium temperature. The largest calculation error will be ignorable when $t = 10\tau$. In general, the relaxation time of matters is very small in physics, which is in the order of *ps*~*fs*, showing that the calculation error will tend to zero very quickly.

For other non-Fourier heat conduction models, including the Jeffrey model, the TT model, and the GK model, there are special relations for physical properties. When the special relations are not satisfied, the largest calculation errors in the whole field still decay with time and reach maximums at $t = 0$, and the maximums are two orders of magnitude smaller than the equilibrium temperature. When the special relations are satisfied, the largest calculation errors in the whole field would not reach maximums at $t = 0$, and the maximums are only one order of magnitude smaller than the equilibrium temperature. Whether or not the special relations are satisfied, the calculation error will be much smaller than T_0 in a short time if the thermal relaxation time is small enough, which is tenable for general non-Fourier heat conduction.

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References

[1] A. N. Smith, J. L. Hostetler, and P. M. Norris, Nonequilibrium Heating in Metal Films: An Analytical and Numerical Analysis, *Numer. Heat Transfer A*, vol. 35, pp. 859–873, 1999.
[2] A. Horvat, Y. Sinai, and P. Tofilo, Semi-Analytical Treatment of Wall Heat Transfer Coupled to a Numerical Simulation Model of Fire, *Numer. Heat Transfer A*, vol. 55, pp. 517–533, 2008.
[3] A. Yevtushenko and S. J. Matysiak, On Approximate Solutions of Temperature and Thermal Stresses in an Elastic Semi-Space Due to Laser Heating, *Numer. Heat Transfer A*, vol. 47, pp. 899–915, 2005.
[4] H. A. Machado and R. M. Cotta, Analysis of Internal Convection with Variable Physical Properties via Integral Transformation, *Numer. Heat Transfer A*, vol. 36, pp. 699–724, 1999.
[5] J. W. Yang and S. G. Bankoff, Solidification Effects on the Fragmentation of Molten Metal Drops behind a Pressure Shock Wave, *J. Heat Transfer*, vol. 109, pp. 226–231, 1987.
[6] K. T. Yang and A. Szewczyk, An Approximate Treatment of Unsteady Heat Conduction in Semi-Infinite Solids with Variable Thermal Properties, *J. Heat Transfer*, vol. 81, pp. 251–252, 1959.
[7] B. T. F. Chung and J. S. Hsiao, Heat Transfer with Ablation in a Finite Slab Subjected to Time-Variant Heat Fluxes, *Aiaa J.*, vol. 23, pp. 145–150, 1985.
[8] T. R. Goodman, Application of Integral Methods to Transient Non-Linear Heat Transfer, *Adv. Heat Transfer*, vol. 1, pp. 51–122, 1964.
[9] P. V´An and W. Muschik, Structure of Variational Principles in Nonequilibrium Thermodynamics, *Phys. Rev. E.*, vol. 52, pp. 3584–3590, 1995.

[10] G. Lebon, P. Perzyna, and K. Wilmanski, *Recent Developments in Thermomechanics of Solids*, pp. 221–366, Springer, Berlin, 1980.
[11] L. Onsager, Reciprocal Relations in Irreversible Processes. I, *Phys. Rev.*, vol. 37, pp. 405–426, 1931.
[12] L. Onsager, Reciprocal Relations in Irreversible Processes. II, *Phys. Rev.*, vol. 38, pp. 2265–2279, 1931.
[13] L. Onsager and S. Maclup, Fluctuations and Irreversible Processes, *Phys. Rev.*, vol. 91, pp. 1505–1512, 1953.
[14] P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations*, pp. 126–219, Wiley, New York, 1971.
[15] M. A. Biot, Variational Principles in Irreversible Thermodynamics with Application to Viscoelasticity, *Phys. Rev.*, vol. 97, pp. 1463–1469, 1955.
[16] M. A. Biot, *Variational Principles in Heat Transfer*, pp. 1–50, Clarendon Press, Oxford University Press, London, 1970.
[17] I. Gyarmati, On the "Governing Principle of Dissipative Processes" and Its Extension to Non-Linear Problems, *Annalen Der Physik*, vol. 23, pp. 353–378, 1969.
[18] J. Verh´as, Gyarmati's Variational Principle of Dissipative Processes, *Entropy*, vol. 16, pp. 2362–2383, 2014.
[19] V. A. Cimmelli, Weakly Nonlocal Thermodynamics of Anisotropic Rigid Heat Conductors Revisited, *J. Non-Equilib. Thermodyn.*, vol. 36, pp. 285–309, 2011.
[20] W. Muschik, P. V´an, and C. Papenfuss, Variational Principles in Thermodynamics, *Technische Mechanik*, vol. 20, pp. 105–112, 2005.
[21] J. Merker and M. Krueger, On a Variational Principle in Thermodynamics, *Contin. Mech. Thermodyn.*, vol. 25, pp. 779–793, 2013.
[22] B. Vujanovic and D. Djukic, On One Variational Principle of Hamilton's Type for Nonlinear Heat Transfer Problem, *Int. J. Heat Mass Transfer*, vol. 15, pp. 1111–1123, 1972.
[23] P. Rosen, On Variational Principles for Irreversible Processes, *J. Chem. Phys.*, vol. 21, pp. 1220–1221, 1953.
[24] A. Prasad and H. C. Agrawal, Biot's Variational Principle for a Stefan Problem, *AIAA J.*, vol. 10, pp. 325–327, 1972.
[25] A. Prasad and S. N. Sinha, Radiative Ablation of Melting Solids, *AIAA J.*, vol. 14, pp. 1494–1497, 1975.
[26] V. D. Rao, P. K. Sarma, and G. J. V. J. Raju, Biot's Variational Method to Fluidized-Bed Coating on Thin Plates, *J. Heat Transfer*, vol. 107, pp. 258–260, 1985.
[27] M. A. Biot, New Methods in Heat Flow Analysis with Application to Flight Structures, *J. Aeronaut. Sci.*, vol. 24, pp. 857–873, 1957.
[28] A. Stark, Approximation Methods for the Solution of Heat Conduction Problems Using Gyarmati's Principle, *Ann. Phys.*, vol. 31, pp. 53–75, 1974.
[29] P. Singh, The Application of the Governing Principle of Dissipative Processes to Benard Convection, *Int. J. Heat Mass Transfer*, vol. 19, pp. 581–588, 1976.
[30] N. H. Tung, An Application of the Governing Principle of Dissipative Processes to Eigenvalue Problem of Hydro-Thermodynamics Fluids, *Acta Phys. Hung.*, vol. 65, pp. 69–78, 1989.
[31] P. Singh, Lagrangian Thermodynamics as a Particular Case of Governing Principle of Dissipative Processes, *Acta Phys. Acad. Sci. Hung.*, vol. 43, pp. 133–140, 1977.
[32] D. D. Joseph and L. Preziosi, Heat Waves, *Rev. Modern Phys.*, vol. 61, pp. 41–73, 1989.
[33] D. S. Tang, Y. C. Hua, and B. Y. Cao, Thermal Wave Propagation Through Nanofilms in Ballistic-Diffusive Regime by Monte Carlo Simulations, *Int. J. Therm. Sci.*, vol. 109, pp. 81–89, 2016.
[34] M. K. Zhang, B. Y. Cao, and Y. C. Guo, Numerical Studies on Damping of Thermal Waves, *Int. J. Therm. Sci.*, vol. 84, pp. 9–20, 2014.
[35] W. J. Yao and B. Y. Cao, Thermal Wave Propagation in Graphene by Molecular Dynamics Simulations, *Chin. Sci. Bulletin*, vol. 59, pp. 3495–3503, 2014.
[36] Y. Dong, B. Y. Cao, and Z. Y. Guo, Generalized Heat Conduction Laws Based on Thermomass Theory and Phonon Hydrodynamics, *J. Appl. Phys.*, vol. 110, pp. 063504, 2011.
[37] B. Y. Cao and Z. Y. Guo, Equation of Motion of Phonon Gas and Non-Fourier Heat Conduction, *J. Appl. Phys.*, vol. 102, pp. 053503, 2007.
[38] J. T. O'Toole, Variational Principles for Time-Dependent Transport Problems, *Chem. Eng. Sci.*, vol. 22, pp. 313–318, 1967.
[39] L. Davies and D. Brian, *Integral Transforms and Their Applications,* Chap. 7, Springer, New York, 2002.
[40] S. N. Li and B. Y. Cao, Generalized Variational Principles for Heat Conduction Models Based on Laplace Transforms, *Int. J. Heat Mass Transfer*, vol. 103, pp. 1176–1180, 2016.
[41] C. Cattaneo, Sur une forme de léquation de lachaleur éliminant le paradoxe d'une propagation instantanée, *Comptes Rendus*, vol. 247, pp. 431–433, 1958.
[42] P. Vernotte, Les Paradoxes De La Théorie Continue De L'équation De La Chaleur, *Comptes Rendus*, vol. 246, pp. 3154–3155, 1958.
[43] S. I. Anisinov, B. L. Kapeliovich, and T. L. Perelman, Electron Emission from Metal Surfaces Exposed to Ultrashort Laser Pulses, *Soviet Phys. JETP*, vol. 39, pp. 375–377, 1974.
[44] R. A. Guyer and J. A. Krumhansl, Solution of the Linearized Phonon Boltzmann Equation, *Phys. Rev.*, vol. 14, pp. 766–778, 1996.